

EXTENDING THE PARKING SPACE

ANDREW BERGET AND BRENDON RHOADES

ABSTRACT. The action of the symmetric group S_n on the set Park_n of parking functions of size n has received a great deal of attention in algebraic combinatorics. We prove that the action of S_n on Park_n extends to an action of S_{n+1} . More precisely, we construct a graded S_{n+1} -module V_n such that the restriction of V_n to S_n is isomorphic to Park_n . We describe the S_n -Frobenius characters of the module V_n in all degrees and describe the S_{n+1} -Frobenius characters of V_n in extreme degrees. We give a bivariate generalization $V_n^{(\ell, m)}$ of our module V_n whose representation theory is governed by a bivariate generalization of Dyck paths. A Fuss generalization of our results is a special case of this bivariate generalization.

1. INTRODUCTION

This paper is about extending the visible permutation action of S_n on the space Park_n spanned by parking functions of size n to a hidden action of the larger symmetric group S_{n+1} . The S_{n+1} -module we construct will be a subspace of the coordinate ring of the reflection representation of type A_n and will inherit the polynomial grading of this coordinate ring. Using statistics on Dyck paths, Theorem 2 will give an explicit combinatorial formula for the graded S_n -Frobenius character of our module. In Theorem 5 we will describe the extended S_{n+1} action in extreme degrees.

As far as the authors know, this is the first example of an extension of the S_n -module structure on Park_n to S_{n+1} and the first proof that the S_n -module structure on Park_n extends to S_{n+1} . Our main theorems should be thought of as parallel to several well known extensions, most notably the action of S_{n+1} on the multilinear subspace of the free Lie algebra on $n+1$ symbols, which extends the regular representation of S_n to S_{n+1} . See Section 4 for more on such questions.

We remark that our result is the ‘best possible’ in two senses. First, it is not always possible to extend Park_n to an S_{n+2} -module; for example, the action of S_4 on Park_4 does not extend to an action of S_6 . Also, from a combinatorial point of view, one may be interested in extending the action of S_n on Park_n to a *permutation* action of the larger symmetric group S_{n+1} . However, it is impossible to extend the action of S_4 on Park_4 to a permutation action of S_5 .

Key words and phrases. parking functions, symmetric group, Dyck paths, representation, matroid.

2. BACKGROUND AND MAIN RESULTS

A length n sequence (a_1, \dots, a_n) of positive integers is called a *parking function of size n* if its nondecreasing rearrangement $(b_1 \leq \dots \leq b_n)$ satisfies $b_i \leq i$ for all i ¹. Parking functions were introduced by Konheim and Weiss [KW] in the context of computer science, but have seen much application in algebraic combinatorics with connections to Catalan combinatorics, Shi hyperplane arrangements, diagonal coinvariant rings, and rational Cherednik algebras. The set of parking functions of size n is famously counted by $(n+1)^{n-1}$. The \mathbb{C} -vector space Park_n spanned by the set of parking functions of size n carries a natural permutation action of the symmetric group S_n on n letters:

$$(2.1) \quad w.(a_1, \dots, a_n) = (a_{w(1)}, \dots, a_{w(n)})$$

for $w \in S_n$ and $(a_1, \dots, a_n) \in \text{Park}_n$.

A *partition* λ of a positive integer n is a weakly decreasing sequence $\lambda = (\lambda_1 \geq \dots \geq \lambda_k)$ of nonnegative integers which sum to n . We write $\lambda \vdash n$ to mean that λ is a partition of n and define $|\lambda| := n$. We call k the *length* of the partition λ . The *Ferrers diagram* of λ consists of λ_i left justified boxes in the i^{th} row from the top ('English notation'). If λ is a partition, we define a new partition $\text{mult}(\lambda)$ whose parts are obtained by listing the (positive) part multiplicities in λ in weakly decreasing order. For example, we have that $\text{mult}(4, 4, 3, 3, 3, 1, 0, 0) = (3, 2, 2, 1)$.

We will make use of two partial orders on partitions in this paper. The first partial order is *Young's lattice* with relations given by $\lambda \subseteq \mu$ if $\lambda_i \leq \mu_i$ for all $i \geq 1$ (where we append an infinite string of zeros to the ends of λ and μ so that these inequalities make sense). Equivalently, we have that $\lambda \subseteq \mu$ if and only if the Ferrers diagram of λ fits inside the Ferrers diagram of μ . *Dominance order* on partitions is defined by $\lambda \preceq \mu$ if for all $i \geq 1$ we have the inequality of partial sums $\lambda_1 + \dots + \lambda_i \leq \mu_1 + \dots + \mu_i$ (where we again append an infinite string of zeros to the ends of λ and μ). Observe that either of the relations $\lambda \subseteq \mu$ or $\lambda \preceq \mu$ imply that $|\lambda| \leq |\mu|$.

For a partition $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$, we let S_λ denote the *Young subgroup* $S_{\lambda_1} \times \dots \times S_{\lambda_k}$ of S_n . We denote by M^λ the coset representation of S_n given by $M^\lambda := \text{Ind}_{S_\lambda}^{S_n}(\mathbf{1}_{S_\lambda}) \cong_{S_n} \mathbb{C}S_n/S_\lambda$ and we denote by S^λ the irreducible representation of S_n labeled by the partition λ .

Let R_n denote the \mathbb{C} -vector space of class functions $S_n \rightarrow \mathbb{C}$. Identifying modules with their characters, the set $\{S^\lambda : \lambda \vdash n\}$ forms a basis of R_n . The graded vector space $R := \bigoplus_{n \geq 0} R_n$ attains the structure of a \mathbb{C} -algebra

¹The terminology arises from the following situation. Consider a linear parking lot with n parking spaces and n cars that want to park in the lot. For $1 \leq i \leq n$, car i wants to park in the space a_i . At stage i of the parking process, car i parks in the first available spot $\geq a_i$, if any such spots are available. If no such spots are available, car i leaves the lot. The driver preference sequence (a_1, \dots, a_n) is a parking function if and only if all cars are able to park in the lot.

via the induction product $S^\lambda \circ S^\mu := \text{Ind}_{S_n \times S_m}^{S_{n+m}} (S^\lambda \otimes_{\mathbb{C}} S^\mu)$, where $\lambda \vdash n$ and $\mu \vdash m$.

We denote by Λ the ring of symmetric functions (in an infinite set of variables X_1, X_2, \dots , with coefficients in \mathbb{C}). The \mathbb{C} -algebra Λ is graded and we denote by Λ_n the homogeneous piece of degree n . Given a partition λ , we denote the corresponding Schur function by s_λ and the corresponding complete homogeneous symmetric function by h_λ .

The *Frobenius character* is the graded \mathbb{C} -algebra isomorphism $\text{Frob} : R \rightarrow \Lambda$ induced by setting $\text{Frob}(S^\lambda) = s_\lambda$. It is well known that we have $\text{Frob}(M^\lambda) = h_\lambda$. Generalizing slightly, if $V = \bigoplus_{k \geq 0} V(k)$ is a graded S_n -module, define the *graded Frobenius character* $\text{grFrob}(V; q) \in \Lambda \otimes_{\mathbb{C}} \mathbb{C}[[q]]$ to be the formal power series in q with coefficients in Λ given by $\text{grFrob}(V; q) := \sum_{k \geq 0} \text{Frob}(V(k))q^k$.

A *Dyck path of size n* is a lattice path D in \mathbb{Z}^2 consisting of vertical steps $(0, 1)$ and horizontal steps $(1, 0)$ which starts at $(0, 0)$, ends at (n, n) , and stays weakly above the line $y = x$. A maximal contiguous sequence of vertical steps in D is called a *vertical run* of D .

We will associate two partitions to a Dyck path D of size n . The *vertical run partition* $\lambda(D) \vdash n$ is obtained by listing the (positive) lengths of the vertical runs of D in weakly decreasing order. For example, if D is the Dyck path in Figure 1, then $\lambda(D) = (3, 2, 1)$. The *area partition* $\mu(D)$ is the partition of length n whose Ferrers diagram is the set of boxes to the upper left of D in the $n \times n$ square with lower left coordinate at the origin. For example, if D is the Dyck path of size 6 in Figure 1, then $\mu(D) = (5, 1, 1, 1, 0, 0)$. The boxes in the Ferrers diagram of $\mu(D)$ are shaded. We define the *area statistic*² on Dyck paths by $\text{area}(D) = |\mu(D)|$. For the Dyck path in our running example, $\text{area}(D) = 8$. By construction, we have that $\text{mult}(\mu(D)) = \lambda(D)$ for any Dyck path D of size n .

Dyck paths of size n can be used to obtain a decomposition of Park_n as a direct sum of coset modules M^λ . In particular, let D be a Dyck path of size n . A *labeling* of D assigns each vertical run of D to a subset of $[n] := \{1, 2, \dots, n\}$ of size equal to the length of that vertical run such that every letter in $[n]$ appears exactly once as a label of a vertical run. Figure 1 shows an example of a labeled Dyck path of size 6, where the subsets labeling the vertical runs are placed just to the right of the runs.

The set of labeled Dyck paths of size n carries an action of S_n given by label permutation. There is an S_n -equivariant bijection from the set of labeled Dyck paths D of size n to parking functions (a_1, \dots, a_n) of size n given by letting a_i be one greater than the x -coordinate of the vertical run of D labeled by i . For example, the labeled Dyck path in Figure 1 corresponds to the parking function $(2, 6, 1, 2, 1, 2) \in \text{Park}_6$. Since any fixed labeled Dyck path of size D generates a cyclic S_n -module isomorphic to

²Many authors instead define the area of a Dyck path D to be the number of complete lattice squares between D and the line $y = x$, so that our statistic would be the ‘coarea’.

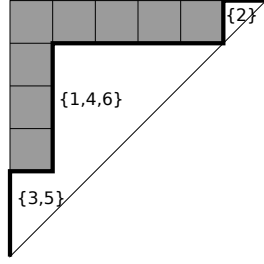


FIGURE 1. A Dyck path of size 6.

$M^{\lambda(D)}$, it is immediate that the parking space Park_n decomposes into coset representations as

$$(2.2) \quad \text{Park}_n \cong_{S_n} \bigoplus_D M^{\lambda(D)},$$

where the direct sum is over all Dyck paths D of size n . Equivalently, we have that the Frobenius character of Park_n is given by $\text{Frob}(\text{Park}_n) = \sum_D h_{\lambda(D)}$. For example, the 5 Dyck paths of size 3 shown in Figure 3 lead to the Frobenius character

$$(2.3) \quad \text{Frob}(\text{Park}_3) = h_{(3)} + 3h_{(2,1)} + h_{(1,1,1)}.$$

The vector space underlying the S_{n+1} -module which will extend Park_n is a subspace of the polynomial ring $\mathbb{C}[x_1, \dots, x_{n+1}]$ in $n+1$ variables and first studied in the work of Postnikov and Shapiro [PoSh]. Let K_{n+1} denote the complete graph on the vertex set $[n+1]$. Given an edge $e = (i < j)$ in K_{n+1} , we associate the polynomial weight $p(e) := x_i - x_j \in \mathbb{C}[x_1, \dots, x_{n+1}]$. A subgraph $G \subseteq K_{n+1}$ (identified with its edge set) gives rise to the polynomial weight $p(G) := \prod_{e \in G} p(e)$. Following Postnikov and Shapiro, we call a subgraph $G \subseteq K_{n+1}$ *slim* if the complement edge set $K_{n+1} - G$ is a connected graph on the vertex set $[n+1]$.

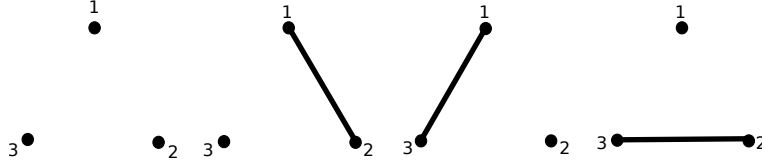
Definition 1. Denote by V_n the \mathbb{C} -linear subspace of $\mathbb{C}[x_1, \dots, x_{n+1}]$ given by

$$(2.4) \quad V_n := \text{span}\{p(G) : G \text{ is a slim subgraph of } K_{n+1}\}.$$

Let $V_n(k)$ denote the homogeneous piece of V_n of polynomial degree k ; the space $V_n(k)$ is spanned by those polynomials $p(G)$ corresponding to slim subgraphs G of K_{n+1} with k edges.

While the set of polynomials $\{p(G) : G \text{ is a slim subgraph of } K_{n+1}\}$ is linearly dependent in general, a basis for V_n can be constructed using standard matroid theoretic results [PoSh, Proposition 9.4]. Fix a total order on the edge set of K_{n+1} . Given a spanning tree T of K_{n+1} , the *external activity* $\text{ex}(T)$ of T is the set of edges $e \in K_{n+1}$ such that e is the minimal edge of the unique cycle in $T \cup \{e\}$. A basis of V_n is given by

$$\{p(K_{n+1} - (\text{ex}(T) \cup T)) : T \text{ is a spanning tree of } K_{n+1}\}.$$

FIGURE 2. The four slim subgraphs of K_3 .

It follows immediately from Cayley's theorem that $\dim V_n = (n+1)^{n-1}$. Aside from this dimension formula, we will make no further use of this basis (or, indeed, any explicit basis) of V_n for the rest of the paper.

Since the slimness of a subgraph is preserved under the action of S_{n+1} on the vertex set $[n+1]$ and $p(G)$ is homogeneous of degree equal to the number of edges in G , it follows that $V_n = \bigoplus_{k \geq 0} V_n(k)$ is a graded S_{n+1} -submodule of the polynomial ring $\mathbb{C}[x_1, \dots, x_{n+1}]$. In fact, the space V_n sits inside the copy of the coordinate ring of the reflection representation of type A_n sitting inside $\mathbb{C}[x_1, \dots, x_{n+1}]$ generated by $x_i - x_{i+1}$ for $1 \leq i \leq n$.

The following result was conjectured by the first author. We postpone its proof, along with the proofs of the other results in this section, to Section 3.

Theorem 2. *Embed S_n into S_{n+1} by letting S_n act on the first n letters. We have that*

$$(2.5) \quad \text{Res}_{S_n}^{S_{n+1}}(V_n(k)) \cong_{S_n} \bigoplus_D M^{\lambda(D)},$$

where the direct sum is over all Dyck paths of size n and area k . In particular, by Equation 2.2 we have that

$$(2.6) \quad \text{Res}_{S_n}^{S_{n+1}}(V_n) \cong_{S_n} \text{Park}_n.$$

Example 3. In the case $n = 2$, Figure 2 shows that four slim subgraphs of the complete graph K_3 . From left to right, the corresponding polynomials are $1, x_1 - x_2, x_1 - x_3$, and $x_2 - x_3$. It follows that $V_2(0) = \text{span}\{1\}$ and $V_2(1) = \text{span}\{x_1 - x_2, x_1 - x_3, x_2 - x_3\}$. Observe that the graded Frobenius character of V_2 is $\text{grFrob}(V_2; q) = s_{(3)}q^0 + s_{(2,1)}q^1$. By the branching rule for symmetric groups (see [Sag]), we have that $\text{grFrob}(\text{Res}_{S_2}^{S_3}(V_2); q) = s_{(2)}q^0 + (s_{(2)} + s_{(1,1)})q^1$. Setting $q = 1$ yields $\text{Frob}(\text{Res}_{S_2}^{S_3}(V_2)) = 2s_{(2)} + s_{(1,1)}$, which agrees with the Frobenius character of Park_2 .

Example 4. Below we have the graded Frobenius character for V_3 and V_4 .

$$\text{grFrob}(V_3) = s_{(4)} + s_{(3,1)}q + (s_{(4)} + s_{(3,1)} + s_{(2,2)})q^2 + (s_{(3,1)} + s_{(2,1,1)})q^3$$

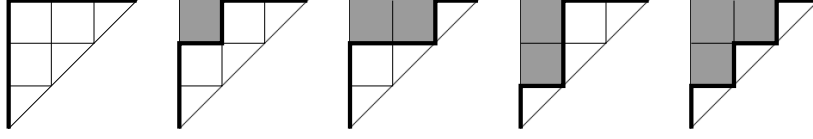


FIGURE 3. The 5 Dyck paths of size 3. From left to right, their contributions to the graded Frobenius character $\text{grFrob}(\text{Res}_{S_3}^{S_4}(V_3); q)$ are $h_{(3)}q^0$, $h_{(2,1)}q^1$, $h_{(2,1)}q^2$, $h_{(2,1)}q^2$, and $h_{(1,1,1)}q^3$.

$$\begin{aligned} \text{grFrob}(V_4) = & s_{(5)} + s_{(4,1)}q + (s_{(5)} + s_{(4,1)} + s_{(3,2)})q^2 \\ & + (s_{(5)} + 2s_{(4,1)} + s_{(3,2)} + s_{(3,1,1)})q^3 \\ & + (s_{(5)} + 2s_{(4,1)} + 2s_{(3,2)} + s_{(3,1,1)} + s_{(2,2,1)})q^4 \\ & + (s_{(5)} + 2s_{(4,1)} + 2s_{(3,2)} + 2s_{(3,1,1)} + s_{(2,2,1)})q^5 \\ & + (s_{(4,1)} + s_{(3,2)} + s_{(3,1,1)} + s_{(2,2,1)} + s_{(2,1,1,1)})q^6. \end{aligned}$$

We leave it to the reader to check that the restrictions of these graded Frobenius characters yield the representations Park_3 and Park_4 of S_3 and S_4 , respectively.

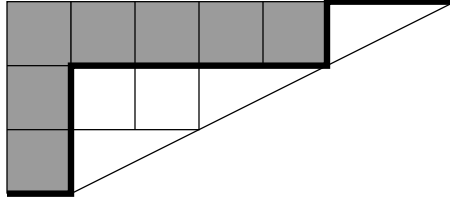
Equivalently, we have that $\text{grFrob}(\text{Res}_{S_n}^{S_{n+1}}(V_n); q) = \sum_D q^{\text{area}(D)} h_{\lambda(D)}$, where the sum is over all Dyck paths D of size n . For example, computing the area and run partitions of the 5 Dyck paths of size 3 shown in Figure 3 shows that

$$(2.7) \quad \text{grFrob}(\text{Res}_{S_3}^{S_4}(V_3); q) = h_{(3)}q^0 + h_{(2,1)}q^1 + 2h_{(2,1)}q^2 + h_{(1,1,1)}q^3.$$

Postnikov and Shapiro showed that the dimension of the vector space V_n is equal to $(n+1)^{n-1}$, however the S_n -module structure of V_n has remained unstudied. Indeed, Theorem 2 is the first description of the S_n -module structure of V_n .

It is natural to ask for an explicit description of the S_{n+1} -structure of V_n or of its graded pieces $V_n(k)$. This problem is open in general, but we can describe the extended structure of $V_n(k)$ in the extreme degrees $k = 0, 1, \dots, n-1$ as well as $k = \binom{n}{2}$. Let C_{n+1} be the cyclic subgroup of S_{n+1} generated by the long cycle $c := (1, 2, \dots, n+1)$ and let ζ be the linear representation of C_{n+1} which sends c to $e^{\frac{2\pi i}{n+1}}$. Mackey's Theorem can be used to prove that the Lie representation $\text{Lie}_n := \text{Ind}_{C_{n+1}}^{S_{n+1}}(\zeta)$ of S_{n+1} satisfies $\text{Res}_{S_n}^{S_{n+1}}(\text{Lie}_n) \cong_{S_n} \mathbb{C}[S_n]$. Stanley proved that the Lie representation arises as the action of S_{n+1} on the top poset cohomology of the lattice of set partitions of $[n+1]$, tensored with the sign representation [St].

Theorem 5. *The module $V_n(0)$ carries the trivial representation of S_{n+1} , the module $V_n(1)$ carries the reflection representation of S_{n+1} , and in general*

FIGURE 4. A $(2, 2)$ -Dyck path of size 3.

$V_n(k) = \text{Sym}^k(V_n(1))$ for $k < n$. The module $V_n(\binom{n}{2}) = V_n(\text{top})$ carries the Lie representation of S_{n+1} tensor the sign representation.

The first part of this result is optimal in the sense that if $k \geq n$ then $V_n(k)$ is a proper subspace of $\text{Sym}^k(V_n(1))$.

We will prove a bivariate generalization of Theorem 2 which includes a ‘Fuss generalization’ as a special case. Given $\ell, m, n > 0$, define a (ℓ, m) -Dyck path of size n to be a lattice path D in \mathbb{Z}^2 consisting of vertical steps $(0, 1)$ and horizontal steps $(1, 0)$ which starts at $(-\ell + 1, 0)$, ends at (mn, n) , and stays weakly above the line $y = \frac{x}{m}$. Taking $\ell = m = 1$, we recover the classical notion of a Dyck path of size n . Taking $\ell = 1$ and m general, the $(1, m)$ -Dyck paths are the natural Fuss extension of Dyck paths. As before, we define the *vertical run partition* $\lambda(D) \vdash n$ of an (ℓ, m) -Dyck path D of size n to be the partition obtained by listing the lengths of the vertical runs of D in weakly decreasing order. We also define the *area partition* $\mu(D)$ to be the length n partition whose Ferrers diagram fits between D and a $(\ell - 1 + mn) \times n$ rectangle with lower left hand coordinate $(-\ell + 1, 0)$. The *area* of D is defined by $\text{area}(D) := |\mu(D)|$. We have that $\text{mult}(\mu(D)) = \lambda(D)$.

Figure 4 shows an example of a $(2, 2)$ -Dyck path of size 3. The path D starts at $(-1, 0)$, ends at $(6, 3)$, and stays above the line $y = \frac{x}{2}$. We have that $\lambda(D) = (2, 1) \vdash 3$, $\mu(D) = (5, 1, 1)$, and $\text{area}(D) = 7$.

Let $K_{n+1}^{(\ell, m)}$ be the multigraph on the vertex set $[n + 1]$ with m edges between i and j for all $1 \leq i < j \leq n$ and ℓ edges between i and $n + 1$ for all $1 \leq i \leq n$. We call a sub-multigraph G of $K_{n+1}^{(\ell, m)}$ *slim* if the multi-edge set difference $K_{n+1}^{(\ell, m)} - G$ is a connected multigraph on $[n + 1]$. We extend the polynomial weight $p(G) \in \mathbb{C}[x_1, \dots, x_{n+1}]$ to multigraphs G in the obvious way.

Definition 6. Let $V_n^{(\ell, m)}$ be the \mathbb{C} -linear subspace of $\mathbb{C}[x_1, \dots, x_{n+1}]$ given by the span

$$(2.8) \quad V_n^{(\ell, m)} := \text{span}\{p(G) : G \text{ is a slim sub-multigraph of } K_{n+1}^{(\ell, m)}\}.$$

As in the case $m = \ell = 1$, the space $V_n^{(\ell, m)}$ is stable under the action of S_n , which respects the grading. When $\ell = m$, $V_n^{(\ell, \ell)}$ also has an S_{n+1} -action, which also respects the grading. Postnikov and Shapiro showed that

the dimension of $V_n^{(\ell,m)}$ is $\ell(mn + \ell)^{n-1}$ [PoSh]. Let $V_n^{(\ell,m)}(k)$ be the degree k piece of $V_n^{(\ell,m)}$.

Theorem 7. *Under action of S_n induced by permutation of vertex labels,*

$$(2.9) \quad V_n^{(\ell,m)}(k) \cong_{S_n} \bigoplus_D M^{\lambda(D)},$$

where the direct sum is over all (ℓ, m) -Dyck paths of size n and area k . The containment $V_n^{(\ell,m)}(k) \subseteq \text{Sym}^k(V_n^{(\ell,m)}(1))$ is an equality for $k < n$.

In the case $m = \ell$, the module $V_n^{(\ell,\ell)}(1)$ has S_{n+1} -structure given by the reflection representation of S_{n+1} , so that the equality $V_n^{(\ell,m)}(k) = \text{Sym}^k(V_n^{(\ell,m)}(1))$, $k < n$, describes the S_{n+1} -module structure completely. The top degree space $V_n^{(\ell,\ell)}(\text{top})$ is $\text{Lie}_n \otimes (\text{sign})^{\otimes \ell}$.

Example 8. Take $n = 3$, $\ell = m = 2$ in Theorem 7, so that $V_3^{(2,2)}$ carries an S_4 action. We have that

$$\begin{aligned} \text{grFrob}(\text{Res}_{S_3}^{S_4}(V_3^{(2,2)})) &= h_{(3)}q^0 + h_{(2,1)}q^1 + 2h_{(2,1)}q^2 + (h_{(3)} + h_{(2,1)} + h_{(1,1,1)})q^3 \\ &\quad + (3h_{(2,1)} + h_{(1,1,1)})q^4 + (3h_{(2,1)} + 2h_{(1,1,1)})q^5 \\ &\quad + (2h_{(2,1)} + 3h_{(1,1,1)})q^6 + (2h_{(2,1)} + 3h_{(1,1,1)})q^7 \\ &\quad + 3h_{(1,1,1)}q^8 + h_{(1,1,1)}q^9. \end{aligned}$$

3. PROOFS

While Theorem 7 implies Theorem 2, the proof of Theorem 7 is a straightforward extension of the proof of Theorem 2 and it will be instructive to prove Theorem 2 first.

The first step in the proof of Theorem 2 is to relate the modules on both sides of the claimed isomorphism by associating a subgraph $G(D)$ of K_{n+1} and a polynomial $p(D) \in \mathbb{C}[x_1, \dots, x_{n+1}]$ to any Dyck path D of size n . We start by labeling the 1×1 box b which is completely above the line $y = x$ with the edge $e(b) = (n - j, n - i)$ in K_{n+1} , where (i, j) is the upper left coordinate of b . See Figure 5 for an example of this labeling in the case $n = 5$. We let $G(D)$ be the subgraph of K_{n+1} consisting of those edges $e(b)$ for which the box b is to the upper left of the path D . In Figure 5, the shaded boxes above the path D each contribute an edge to the subgraph $G(D)$ and we have that $G(D) = \{1 - 6, 1 - 5, 1 - 4, 1 - 3, 2 - 6, 2 - 5, 3 - 6\}$.

Lemma 9. *The subgraph $G(D)$ is slim for any Dyck path D .*

Proof. The subgraph $G(D)$ contains none of the edges in the path $1 - 2 - \dots - (n + 1)$. \square

By Lemma 9, the polynomial $p(D) := p(G(D))$ is contained in V_n . For example, if $n = 5$ and D is the Dyck path shown in Figure 5, we have that

$$(3.1) \quad p(D) = (x_1 - x_6)(x_1 - x_5)(x_1 - x_4)(x_1 - x_3)(x_2 - x_6)(x_2 - x_5)(x_3 - x_6) \in V_5.$$

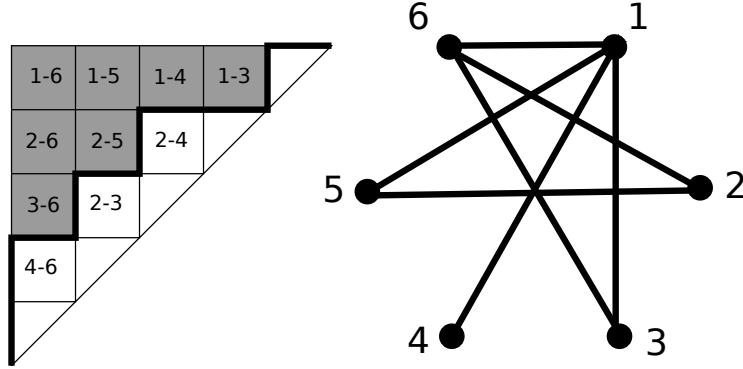


FIGURE 5. A Dyck path D of size 5 and the associated subgraph $G(D)$ of K_6 .

By construction, for any Dyck path D the polynomial $p(D)$ is homogeneous with degree equal to $\text{area}(D)$.

In order to prove the direct sum decomposition in Theorem 2, we will show that the polynomials $p(D)$ project nicely onto a certain subspace of $\mathbb{C}[x_1, \dots, x_{n+1}]$. Since Theorem 2 only concerns the restriction of V_n to S_n , it is natural to consider a subspace of $\mathbb{C}[x_1, \dots, x_{n+1}]$ which is closed under the action of S_n but not of S_{n+1} .

Let $st_n := (n-1, n-2, \dots, 1)$ be the *staircase partition* of length $n-1$. We call a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ *sub-staircase* if $\lambda \subseteq st_n$ (observe that this definition has tacit dependence on n). For any Dyck path D of size n , the partition $\mu(D)$ is sub-staircase.

For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, we use the shorthand $x^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n} \in \mathbb{C}[x_1, \dots, x_n]$. We call a monomial $x_1^{d_1} \cdots x_{n+1}^{d_{n+1}}$ in the variables x_1, \dots, x_{n+1} *sub-staircase* if there exists a permutation $w \in S_n$ and a sub-staircase partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \vdash n$

$$(3.2) \quad x_1^{d_1} \cdots x_{n+1}^{d_{n+1}} = w.x^\lambda.$$

In particular, the variable x_{n+1} does not appear in any sub-staircase monomial. If the monomial $x_1^{d_1} \cdots x_{n+1}^{d_{n+1}}$ is sub-staircase, the partition λ is uniquely determined from the monomial; call this the *exponent partition* of the monomial. Let $W_n \subset \mathbb{C}[x_1, \dots, x_{n+1}]$ be the \mathbb{C} -linear span of all sub-staircase monomials. The subspace W_n is closed under the action of S_n , but not under the action of S_{n+1} .

In the case $n = 3$, the S_3 -orbits of the 16 sub-staircase monomials in $\mathbb{C}[x_1, \dots, x_4]$ are shown in the following table, where the left column shows a representative from each orbit.

1	
x_1	x_2, x_3
x_1^2	x_2^2, x_3^2
$x_1 x_2$	$x_1 x_3, x_2 x_3$
$x_1^2 x_2$	$x_1^2 x_3, x_2^2 x_1, x_2^2 x_3, x_3^2 x_1, x_3^2 x_2$

The S_3 -orbits are parametrized by sub-staircase partitions $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and each orbit contains a unique representative of the form x^λ . The staircase monomials form a linear basis of W_3 and the cyclic S_3 -submodule of W_3 generated by x^λ is isomorphic to $M^{\text{mult}(\lambda)}$. The natural bijection between exponent vectors and parking functions affords an isomorphism $W_3 \cong_{S_3} \text{Park}_3$. These observations generalize in a straightforward way to the following lemma, whose proof is left to the reader.

Lemma 10. *The set of sub-staircase monomials forms a linear basis for W_n and is closed under the action of S_n . The S_n -orbits are parametrized by sub-staircase partitions λ , and the orbit labeled by λ has a unique monomial of the form x^λ . The cyclic S_n -submodule of W_n generated by x^λ is isomorphic to $M^{\text{mult}(\lambda)}$ and we have that $W_n \cong_{S_n} \text{Park}_n$.*

With Lemma 10 in mind, we will construct a graded S_n -module isomorphism $V_n \xrightarrow{\sim} W_n$. We define a graded S_n -module homomorphism $\phi : V_n \rightarrow W_n$ by the following composition:

$$(3.3) \quad \phi : V_n \hookrightarrow \mathbb{C}[x_1, \dots, x_{n+1}] \twoheadrightarrow \mathbb{C}[x_1, \dots, x_n] \twoheadrightarrow W_n,$$

where the first map is inclusion, the second is the specialization $x_{n+1} = 0$, and the third linear map fixes the space W_n pointwise and sends monomials which are not sub-staircase to zero.

We want to show that ϕ is an isomorphism. Postnikov and Shapiro showed that $\dim(W_n) = \dim(V_n) = (n+1)^{n-1}$ [PoSh], so it is enough to show that ϕ is surjective. We will do this by analyzing the polynomials $\phi(p(D))$, where D is a Dyck path of size n .

The set of sub-staircase partitions forms an order ideal in dominance order. The next lemma states that the transition matrix between the set $\{\phi(p(D)) : D \text{ a Dyck path of size } n\}$ expands in the monomial basis of W_n given by $\{x^\lambda : \lambda \text{ sub-staircase}\}$ in a unitriangular way with respect to any linear extension of dominance order (where we associate $\phi(p(D))$ with the partition $\mu(D)$).

Lemma 11. *Let D be a Dyck path of size n . There exist integers $c_{\lambda,w} \in \mathbb{Z}$ such that*

$$(3.4) \quad \phi(p(D)) = x^{\mu(D)} + \sum_{\substack{\lambda \prec \mu(D) \\ |\lambda| = |\mu(D)| \\ w \in S_n}} c_{\lambda,w} w \cdot x^\lambda.$$

Proof. By definition, we have that

$$(3.5) \quad p(D) = \prod_{e=(i<j) \in G(D)} (x_i - x_j),$$

so (up to sign) a typical monomial in the expansion of $p(D)$ is obtained by choosing an endpoint of every edge in $G(D)$ and multiplying the corresponding variables. The map ϕ kills any monomial which contains the variable x_{n+1} , so up to sign a typical monomial in $\phi(p(D))$ is obtained by choosing an endpoint of each edge in $G(D)$ and multiplying the corresponding variables such that whenever $G(D)$ has an edge of the form $(i < n+1)$, we choose the smaller endpoint i . The result follows from the construction of $G(D)$ and the definition of dominance order. \square

As an example of Lemma 11, consider the case $n = 5$ and let the Dyck path D be shown in Figure 5 with $\mu(D) = (4, 2, 1)$. To calculate $\phi(p(D))$, we set $x_6 = 0$ in the product formula for $p(D)$ given in Equation 3.1 and expand. The resulting polynomial is

$$\phi(p(D)) = x_1(x_1 - x_5)(x_1 - x_4)(x_1 - x_3)x_2(x_2 - x_5)x_3 = x_1^4x_2^2x_3 + \cdots$$

where the elipsis denotes terms involving sub-staircase monomials with exponent partition $\prec (4, 2, 1)$. We are ready to complete the proof of Theorem 2.

Proof of Theorem 2. By Lemma 10, the set of sub-staircase monomials forms a linear basis of W_n , so Lemma 11 implies that the S_n -module homomorphism $\phi : V_n \rightarrow W_n$ is surjective. Since $\dim(V_n) = \dim(W_n)$, this implies that ϕ is also injective and gives an isomorphism $\text{Res}_{S_n}^{S_{n+1}}(V_n) \cong_{S_n} \text{Park}_n$. To prove the graded isomorphism in Theorem 2, it is enough to observe that $\text{mult}(\mu(D)) = \lambda(D)$ for any Dyck path D and apply Lemmas 10 and 11 together with the fact that ϕ is graded. \square

It may be tempting to guess that $p(D)$ generates a cyclic S_n -submodule of V_n isomorphic to $M^{\lambda(D)}$, but this is false in general. The reason for this is that while the ‘leading term’ in the expansion of $\phi(p(D))$ in Lemma 11 generates the submodule $M^{\lambda(D)}$ under the action of S_n , the other terms in this expansion can cause $\phi(p(D))$ to generate a different cyclic submodule. For example, consider the case $n = 3$ where D is the Dyck path corresponding to the subgraph of K_4 with two edges $\{1, 3\}, \{1, 4\}$. Then, the S_3 -module generated by either $p(D)$ or $\phi(p(D))$ is 5 dimensional with Frobenius character $s_{(3)} + 2s_{(2,1)}$.

We are ready to prove the claimed S_{n+1} -structure of the extreme degrees of the graded module $V_n(k)$.

Proof of Theorem 5. It is clear from the definitions that $V_n(0)$ carries the trivial representation of S_{n+1} . The space $V_n(1)$ has basis given by the polynomials $x_1 - x_2, x_2 - x_3, \dots, x_n - x_{n+1}$ and hence carries the reflection representation of S_{n+1} (i.e., the irreducible S_{n+1} -module corresponding to the partition $(n, 1)$). Since $V_n \subseteq \text{Sym}(V_n(1))$ we are claiming that in degree

$k < n$ this is an equality. The Hilbert series of V_n is the Tutte polynomial evaluation $q^{\binom{n+1}{2}-n} T_{K_{n+1}}(1, 1/q)$ and so we must prove that the first $n-1$ terms of this sum are the binomial coefficients $\binom{n+k-1}{k}$. There is nothing special about K_{n+1} in this claim and we will prove a more general statement in Lemma 12.

To prove that $V_n(\text{top})$ is isomorphic to $\text{Lie}_{n+1} \otimes \text{sign}$ we reason as follows. The space $V_n(\text{top})$ is spanned by those $p(G)$ where the complementary subgraph $K_{n+1} \setminus G$ is connected and has n edges.

Let \mathcal{A}_n denote the braid arrangement in \mathbb{C}^{n+1} , which is the union of those hyperplanes with at least two coordinates equal. Let $H^*(\mathbb{C}^{n+1} \setminus \mathcal{A}_n; \mathbb{C})$ denote the (complexified) de Rham cohomology ring of its complement. Consider, now, the linear map $c : V_n(\text{top}) \rightarrow H^n(\mathbb{C}^{n+1} \setminus \mathcal{A}_n)$ that sends

$$p(G) \mapsto p(G) \cdot d(x_1 - x_2) \wedge d(x_2 - x_3) \wedge \cdots \wedge d(x_n - x_{n+1}) / \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

This is an isomorphism of vector spaces, since it is division by the Vandermonde product, followed by multiplication by the n -form. To see that c is equivariant notice that $\bigwedge^n V_n(1)$ carries the sign representation of S_{n+1} , because it is 1 dimensional and non-trivial. Likewise does the one dimensional representation spanned by the Vandermonde product. It follows that the signs introduced by multiplication by the n -form and division by the Vandermonde cancel, and c is equivariant.

Finally, the top degree cohomology of the complement $\mathbb{C}^{n+1} \setminus \mathcal{A}_n$ is known to be isomorphic to the top degree *Whitney homology* of its lattice of flats [Bj, Theorem 7.2.10], and this correspondence is at once seen to be S_{n+1} -equivariant. The lattice of flats of \mathcal{A}_n is the partition lattice Π_{n+1} and by a result of Stanley [St] (beautifully recounted by Wachs in [Wa, Section 4.4]), the top degree Whitney homology of the partition lattice Π_{n+1} is $\text{Lie}_{n+1} \otimes \text{sign}$. \square

Lemma 12. *Let G be a connected graph on v vertices with e edges. Denote the Tutte polynomial of G by $T_G(x, y)$. Then, the polynomial $q^{e-v+1} T_G(1, 1/q)$ takes the form,*

$$1 + (v-1)q + \binom{v}{2}q^2 + \binom{v+1}{3}q^3 + \cdots + \binom{(v-1) + (v-2) - 1}{v-2}q^{v-2} + O(q^{v-1}).$$

Proof. We write $T_G(x, y)$ in terms of the two variable *coboundary polynomial*, $\bar{\chi}_G(\lambda, \nu)$. This is the sum

$$\bar{\chi}_G(\lambda, \nu) = \frac{1}{\lambda} \sum_{i=0}^e c_i(G; \lambda) \nu^i$$

where $c_i(G; \lambda)$ is the number of ways to color the vertices of G with λ colors and exactly i monochromatic edges. It is a fact that this is a polynomial in

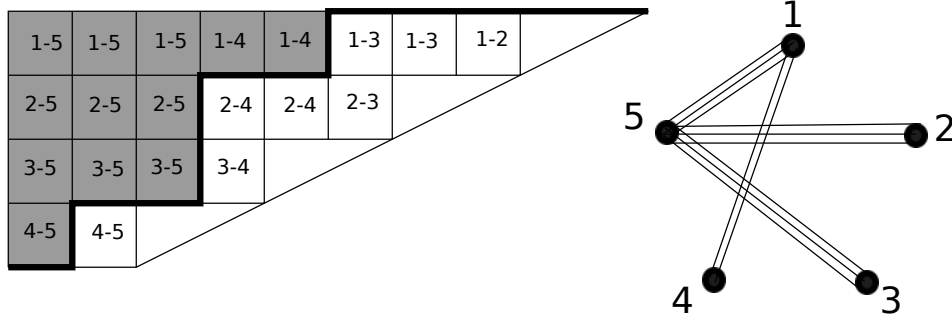


FIGURE 6. A $(3, 2)$ -Dyck path D of size 4 and the associated sub-multigraph $G(D)$ of $K_4^{(3,2)}$.

λ and ν . Now by [BrO, Proposition 6.3.26],

$$q^{e-v+1}T_G(1, 1/q) = \frac{q^e}{(1-q)^{v-1}}\bar{\chi}_G(0, 1/q).$$

Thus, to prove the first part of the lemma we will show that $c_i(G; \lambda) = 0$ for $e - v + 1 < i < e$, and that $c_e(G; \lambda) = \lambda$. Suppose that we have colored the vertices of G and we have more than $e - v + 1$ monochromatic edges. Then the collection of monochromatic edges forms a connected subgraph of G . It follows that all vertices of G are colored the same and hence all edges of G are monochromatic. This means that $c_i(G; \lambda) = 0$ unless $i = e$. That $c_e(G; \lambda) = \lambda$ is clear. \square

The proof of Theorem 7 is a straightforward extension of the proof of Theorem 2. We will be somewhat brief.

Proof of Theorem 7. Given any (ℓ, m) -Dyck path D of size n we associate a sub-multigraph $G(D)$ of $K_{n+1}^{(\ell, m)}$ by letting every box which contributes to $\text{area}(D)$ correspond to a single edge in the multigraph $G(D)$; the labeling which accomplishes this is shown in Figure 6 in the case $(\ell, m) = (3, 2)$ and $n = 4$. For general ℓ, m , and n , we label the boxes in the i^{th} row from the top from left to right with $(\ell + m - 2)$ copies of the edge $i - (n + 1)$, m copies of the edge $i - n$, m copies of the edge $i - (n - 1)$, \dots , m copies of the edge $i - (i + 2)$, and $(m - 1)$ copies of the edge $i - (i + 1)$.

For any (ℓ, m) -Dyck path D of size n , the multigraph complement of $G(D)$ within $K_n^{(\ell, m)}$ contains each of the edges in the path $1 - 2 - \dots - n - (n + 1)$ with multiplicity at least one. Therefore, the sub-multigraph $G(D)$ is slim and the polynomial $p(D) := p(G(D))$ is contained in $V_n^{(\ell, m)}$.

We say that a partition λ with n parts is *sub- (ℓ, m) -staircase* if in Young's lattice we have the relation $\lambda \subseteq (\ell - 1 + m(n - 1), \ell - 1 + m(n - 2), \dots, \ell - 1)$. A monomial $x_1^{d_1} \dots x_{n+1}^{d_{n+1}}$ is *sub- (ℓ, m) -staircase* if there exists $w \in S_n$ and a

sub- (ℓ, m) -staircase partition λ such that

$$(3.6) \quad x_1^{d_1} \cdots x_{n+1}^{d_{n+1}} = x_{w(1)}^{\lambda_1} \cdots x_{w(n)}^{\lambda_n}.$$

In particular, the variable x_{n+1} does not appear in any sub- (ℓ, m) -staircase monomial.

Let $W_n^{(\ell, m)}$ be the subspace of $\mathbb{C}[x_1, \dots, x_{n+1}]$ spanned by the set of all sub- (ℓ, m) -staircase monomials. The space $W_n^{(\ell, m)}$ carries a graded action of S_n . The argument used to prove Lemma 10 extends to show that the degree k homogeneous piece of $W_n^{(\ell, m)}$ is isomorphic as an S_n -module to the direct sum on the right hand side of the isomorphism asserted in Theorem 7.

The isomorphism in Theorem 7 is proven by showing that the graded S_n -module homomorphism $\phi^{(\ell, m)} : V_n^{(\ell, m)} \rightarrow W_n^{(\ell, m)}$ given by the composite

$$(3.7) \quad \phi^{(\ell, m)} : V_n^{(\ell, m)} \hookrightarrow \mathbb{C}[x_1, \dots, x_{n+1}] \rightarrow \mathbb{C}[x_1, \dots, x_n] \rightarrow W_n^{(\ell, m)}$$

is an isomorphism, where the first map is inclusion, the second is the evaluation $x_{n+1} = 0$, and the third fixes $W_n^{(\ell, m)}$ pointwise and sends every monomial which is not sub- (ℓ, m) -staircase to zero.

Postnikov and Shapiro proved that the vector space $V_n^{(\ell, m)}$ has dimension $\ell(\ell + mn)^{n-1}$ [PoSh]. Pitman and Stanley [PiSt] and Yan [Y] showed that the number of (exponent vector of) sub- (ℓ, m) -staircase monomials equals $\ell(\ell + mn)^{n-1}$. Since these monomials form a basis for $W_n^{(\ell, m)}$, in order to prove that $\phi^{(\ell, m)}$ is an isomorphism of S_n -modules, it is enough to show that $\phi^{(\ell, m)}$ is surjective.

The fact that $\phi^{(\ell, m)}$ is surjective follows from the following triangularity result which generalizes Lemma 11. Recall that $\mu(D)$ is the partition whose Ferrers diagram lies to the northwest of an (ℓ, m) -Dyck path D (for example, if D is the $(3, 2)$ -Dyck path appearing on the left in Figure 6, then $\mu(D) = (5, 3, 3, 1)$). The proof of Lemma 13 is almost identical to the proof of Lemma 11 and is left to the reader.

Lemma 13. *Let D be an (ℓ, m) -Dyck path of size n . The monomial expansion of $\phi^{(\ell, m)}(p(D))$ has the form*

$$(3.8) \quad \phi^{(\ell, m)}(p(D)) = x^{\mu(D)} + \cdots,$$

where the ellipsis denotes terms involving monomials whose exponent partitions are $\prec \mu(D)$.

For example, if D is the $(3, 2)$ -Dyck path in Figure 6, then

$$p(D) = (x_1 - x_4)^2(x_1 - x_5)^3(x_2 - x_5)^3(x_3 - x_5)^3(x_4 - x_5)$$

and

$$\phi^{(3, 2)}(p(D)) = x_1^5 x_2^3 x_3^3 x_4^1 + \text{terms whose exponent partitions are } \prec (5, 3, 3, 1).$$

Lemma 13 implies that $\phi^{(\ell, m)}$ is surjective, and dimension counting implies that $\phi^{(\ell, m)}$ is a graded S_n -module homomorphism. The S_n -isomorphism in Theorem 7 follows.

To prove the remainder of Theorem 7, we need to show that $V_n^{(\ell, m)}(k) = \text{Sym}^k(V_n^{(\ell, m)}(1))$ for $k < n$. This follows at once from Lemma 12, since the Hilbert series of $V_n^{(\ell, m)}$ is the Tutte polynomial evaluation

$$q^{\ell \binom{n}{2} + mn - n} T_{K_{n+1}^{(\ell, m)}}(1, 1/q).$$

For this see [PoSh]. For the statement about the top degree, we take the elements in $V_n^{(\ell, \ell)}(\text{top})$, divide them all by $\prod_{1 \leq i < j \leq n+1} (x_i - x_j)^{\ell-1}$, which yields an equivariant isomorphism with $V_n(\text{top}) \otimes \text{sign}^{\otimes(\ell-1)}$. By Theorem 5 this is $\text{Lie}_n \otimes \text{sign}^{\otimes \ell}$. The remainder of Theorem 7 is now proved. \square

4. CONCLUDING REMARKS

In this paper we constructed a graded S_{n+1} -module V_n which satisfies $\text{Res}_{S_n}^{S_{n+1}}(V_n) \cong_{S_n} \text{Park}_n$. While we know the S_{n+1} -structure of V_n in extreme degrees, the full S_{n+1} -structure remains unknown.

Problem 14. *Give a nice expression for the graded S_{n+1} -Frobenius character of V_n .*

Problem 14 may have an answer in terms of free Lie algebras. Let Lie_{n+1} be the free Lie algebra on the generators x_1, \dots, x_{n+1} . The group S_{n+1} acts on Lie_{n+1} by subscript permutation. By keeping track of the multiplicities of the x_i , the S_{n+1} -module Lie_{n+1} carries the structure of an \mathbb{N}^{n+1} -graded vector space. The $(1, \dots, 1)$ -component of this vector space is stable under the action of S_{n+1} and is known to carry the Lie representation, or $V_n(\text{top}) \otimes \text{sign}$. Lower degrees of V_n may also embed naturally inside free Lie algebras.

As we mentioned in Section 1, there does not exist an S_6 -module M such that $\text{Res}_{S_4}^{S_6}(M) \cong_{S_4} \text{Park}_4$, so we cannot hope for an extension of Park_n to a symmetric group of higher rank than $n+1$ in general.

On the other hand, we identified the top degree $V_n(\text{top})$ with the twisted Lie representation $\text{Lie}_n \otimes \text{sign}$ of S_{n+1} . Whitehouse [W] proved that the representation Lie_n extends to S_{n+2} . This suggests the following problem.

Problem 15. *For which values of n and k does $V_n(k)$ extend to a representation of S_{n+2} ?*

By Whitehouse's result, for any $n > 0$, the k -value $k = \binom{n}{2}$ leads to an extension as in Problem 15. Also, since $V_n(0)$ is the trivial representation of S_{n+1} , one can take $k = 0$ and n arbitrary. On the other hand, if $k = 1$ we have that $V_n(1)$ is the reflection representation of S_{n+1} . For $n > 3$, this representation is not the restriction of any S_{n+2} -module.

The results of this paper and that of Whitehouse [W] motivate the following problem which in the opinion of the authors has received surprisingly little attention.

Problem 16. *Let M be an S_n -module. Give a nice criterion for deciding whether M extends to a representation of S_{n+1} .*

Very few irreducible representations of S_n extend to S_{n+1} . Indeed, if S^λ is the irreducible representation of S_n labeled by a partition $\lambda \vdash n$, then S^λ extends to S_{n+1} if and only if λ is a ‘near rectangle’, i.e. a rectangular partition with $n+1$ boxes minus its outer corner.

On the other hand, an ‘asymptotically nonzero fraction’ of S_n -modules extend to S_{n+1} . More precisely, recall that the \mathbb{Z} -module R_n of class functions on $S_n \rightarrow \mathbb{C}$ has basis given by the set of irreducible characters $\{S^\lambda : \lambda \vdash n\}$ (where we identify modules with characters). The \mathbb{Z} -linear map $\psi : R_{n+1} \rightarrow R_n$ induced by restriction is surjective. Indeed, if $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$, form a new partition $\lambda^+ := (\lambda_1 + 1, \lambda_2, \dots, \lambda_k) \vdash n+1$ by increasing the first part of λ by one. By the branching rule for symmetric groups, the restriction $\text{Res}_{S_n}^{S_{n+1}}(S^{\lambda^+})$ has the form

$$\text{Res}_{S_n}^{S_{n+1}}(S^{\lambda^+}) \cong_{S_n} S^\lambda \oplus \dots,$$

where the elipsis denotes a direct sum of irreducibles corresponding to partitions $\succ \lambda$. The surjectivity of ψ follows.

On the level of representations, the fact that ψ is surjective means that the set C_n of S_n -modules which extend to S_{n+1} forms a full rank cone within the integer cone of representations of S_n . One way to interpret Problem 16 would be to describe the extremal rays and/or facets of C_n . Identifying representations of S_n with points in $\mathbb{N}^{p(n)}$, where $p(n) = \#\{\lambda : \lambda \vdash n\}$ is the partition number, we could also ask for the size of C_n by asking for the limit $\lim_{m \rightarrow \infty} \frac{\#(C_n \cap \{0, 1, \dots, m\}^{p(n)})}{(m+1)^{p(n)}}$. The fact that ψ is surjective means that this limit is nonzero, but we have no conjecture as to its value.

Since ψ is surjective, every representation of S_n is a restriction of a virtual S_{n+1} -module. In a slightly different direction, one could ask that a sufficiently large multiple of a representation extend. For arbitrary $m \leq n$, there is some integer M such that $\mathbb{C}[S_m]^{\oplus M}$ extends to an S_{n+1} -module, by a result of Donkin [Do]. Indeed, take the canonical embedding

$$S_m \subset S_{n+1} \subset \text{GL}_{n+1}(\mathbb{C})$$

in the canonical way. Donkin asserts that there is a finite dimensional rational $\text{GL}_{n+1}(\mathbb{C})$ -module V such that $\text{Res}_{S_n}^{\text{GL}_{n+1}(\mathbb{C})} V \approx \mathbb{C}[S_n]^{\oplus m}$. It follows that $\text{Res}_{S_{n+1}}^{\text{GL}_{n+1}(\mathbb{C})} V$ is the desired extension.

Problem 16 is unsolved even for the coset representations M^λ . Not all of these representations extend (if they did, then any direct sum of coset representations such as the parking representation would extend automatically). For example, the representation $M^{(3,2,2)}$ of S_7 does not extend to a representation of S_8 . However, the representation M^λ of S_n extends to a representation of S_{n+1} for all $\lambda \vdash n$ and $0 \leq n \leq 6$.

Even more general than Problem 16, one could ask for a nice way of determining the greatest integer k such that an S_n -module M extends to S_{n+k} . Also, one could ask whether a given *permutation* representation of S_n

extends to S_{n+1} (for example, the parking space extends as a representation, but not as a permutation representation). This problem could also be interesting in positive characteristic or for towers of linear or Weyl groups other than symmetric groups.

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REFERENCES

- [Bj] A. Björner, *The homology and shellability of matroids and geometric lattices* in Matroid Applications (ed. N. White), Encyclopedia of Mathematics and Its Applications, **40**, Cambridge Univ. Press 1992.
- [BrO] T. Brylawski and J. G. Oxley, *The Tutte polynomial and its applications*, in Matroid Applications (ed. N. White), Encyclopedia of Mathematics and Its Applications, **40**, Cambridge Univ. Press 1992.
- [Do] S. Donkin, On free modules for finite subgroups of algebraic groups, *J. London Math. Soc.* (2) **55** (1997), no. 2, 287–296.
- [KW] A. G. Konheim and B. Weiss, An occupancy discipline and applications, *SIAM J. Applied Math.* **14** (1966), 1266–1274.
- [PiSt] J. Pitman and R. Stanley, A polytope related the empirical distributions, plane trees, parking functions, and the associahedron, *Discrete Comput. Geom.* **27** (2002), 603–634.
- [PoSh] A. Postnikov and B. Shapiro, Trees, parking functions, syzygies, and deformations of monomial ideals, *Trans. Amer. Math. Soc.* **356** (2004), 3109–3142.
- [Sag] B. Sagan, *The Symmetric Group*, New York: Springer-Verlag, 2000.
- [St] R. Stanley, Some aspects of groups acting on finite posets, *J. Comb. Theory Ser. A* **32** (1982), 132–161.
- [Wa] M. Wachs, Poset topology: tools and applications. In *Geometric combinatorics*, volume 13 of the *IAS/Park City Math. Ser.*, 497–615. Amer. Math. Soc., Providence, RI, 2007.
- [W] S. Whitehouse, The Eulerian representations of Σ_n as restrictions of representations of Σ_{n+1} , *J. Pure Appl. Algebra* **115** (1997), 309–320.
- [Y] C. H. Yan, On the enumeration of generalized parking functions, Proceedings of the 31st Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 2000), *Congressus Numerantium* **147** (2000), 201–209.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WA, USA
E-mail address: `aberget@math.washington.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO, LA JOLLA, CA, USA
E-mail address: `bprhoades@math.ucsd.edu`